# Calculus - Derivatives

Juho Välimäki — https://vlimki.dev

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## 1 Basics

### 1.1 Definitions

- The **derivative** of a function $f$, denoted $\frac{d}{dx}f(x)$ or $f'(x)$ is essentially the instantenous rate of change of $f$ at $x$.
- The official definition for the derivative $\frac{df}{dx}$ is $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.
- If $f$ has a derivative at $x = c$, $f$ is continuous at $x = c$.
- The **second derivative**, denoted $f''(x)$ or $\frac{d^2f}{dx^2}$ of a function is the derivative of its derivative. If a function is differentiable, that means its derivative is also differentiable.
- In the same way, you can denote the $n$ th derivative of a function as $\frac{d^ny}{dx^n}$.

**Example 1.** Evaluate $\frac{d}{dx}f(x)$, where $f(x) = x^2$

**Solution:** $\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2-x^2}{h}$
$= \lim_{h \to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} 2x + h = 2x$

**Example 2.** Evaluate $\frac{dp}{dq}$, where $p = q^{3/2}$

**Solution:** Knowing that $\frac{d}{dx}x^n = nx^{n-1}$, we know that $\frac{dp}{dq} = \frac{3}{2}q^{3/2-1} = \frac{3}{2}q^{1/2} = \frac{3}{2}\sqrt{q}$

## 1.2 Rules

- The **sum rule** states that the derivative of a sum $\frac{d}{dx}f(x)+g(x)$ is $f'(x)+g'(x)$.
- The **difference rule** is an extension of the sum rule that states that the derivative of a difference is the difference of their derivatives. $\frac{d}{dx}f(x)-g(x)=\frac{d}{x}f(x)+(-g(x))=f'(x)+(-g('x))=f'(x)-g'(x)$.
- The **product rule** states that the derivative of a product $\frac{d}{dx}f(x)\cdot g(x)$ is $f(x)\cdot g'(x)+f'(x)\cdot g(x)$.
- The **quotient rule** states that the derivative of a quotient $\frac{d}{dx}\frac{f(x)}{g(x)}$ is $\frac{g'(x)\cdot f(x)-f'(x)\cdot g(x)}{g^2(x)}$.
- The derivative for exponential functions of form $a^x$ is $L*a^x$ where L is some constant. $L$ is the derivative of $a^x$ at $x=0$.
- For the natural exponential function $e^x$, L is 1. So the derivative of $e^x$ is $e^x$.

## 1.3 Derivatives and Instantaneous Rate of Change

- Derivatives can be utilized in physics for example, where you're being asked the velocity of an object at a particular point in time.
- Consider a function $s$ (distance) of time $t$. To get the speed $v$ at any instantaneous point in time, you use the derivative $\frac{ds}{dt}$. You check how much the distance changes in relation to time for a certain interval.
- The definition of **velocity (instantenous velocity)** is given as $\frac{ds}{dt}$.
- The formula for free-fall distance is defined as $s=\frac{1}{2}gt^2$, where $g=9.8m/s^2$. So this gets us to $4.9t^2$. Here we can use the derivative of $s$, $\frac{ds}{dt}=9.8t$.
- So after an object has fallen for 10 seconds ($t=10$), the velocity of it will be equal to $9.8\cdot 10^2=980m/s$
- **Acceleration** is defined as the derivative of velocity. $\frac{d}{dt}9.8t=9.8m/s$.

## 1.4 Derivatives of Trigonometric Functions

- $\frac{d}{dx}\sin(x)=\cos(x)$
- $\frac{d}{dx}\cos(x)=-\sin(x)$

## 1.5 The Chain Rule

- **The chain rule allows** for deriving function compositions $(f \circ g)(x)$.
- **The power chain rule** extends the chain rule, such that $\frac{d}{dx}u^n = nu^{n-1} \cdot \frac{dx}{du}$
- Consider a function $y = f(u)$ and a function $u = f(x)$. The derivative $\frac{dy}{dx}$ is given by $\frac{du}{dx} \cdot \frac{dy}{du}$.

**Example 1.** Evaluate the derivative of $y = (3x^2 + 1)^2$.

**Solution:** The function is composed of two functions: $f(g) = g^2$ and $g(x) = 3x^2 + 1$. The derivative $\frac{df}{dx}$ will be given by $\frac{df}{dg} \cdot \frac{dg}{dx}$. Considering that $\frac{dg}{dx} = 6x$ and $\frac{df}{dg} = 2g$, the derivative $\frac{df}{dx} = 2(3x^2 + 1) \cdot 6x = 36x^3 + 12x$.

**Example 2.** Evaluate the derivative of $y = e^{cos^2(\pi t - 1)}$.

**Solution:** f(x) is composed of multiple functions, $e^u$, $u = y^2$, $y = cos(z)$, $z = \pi t - 1$.

$$\frac{dy}{dt} = \frac{d}{dt}e^{cos^2(\pi t-1)} \cdot \frac{d}{dt}cos(\pi t - 1) \cdot \frac{d}{dx}cos^2(\pi t - 1)$$
$$= e^{cos^2(\pi t-1)} \cdot (-sin(\pi t - 1)) \cdot \pi \cdot 2 \cdot cos(\pi t - 1)$$

## 1.6 Derivatives of Inverse Functions and Logarithms

- The derivative of the inverse function of $f$, $(f^{-1})'(x)$ is $\frac{1}{f'(f^{-1}(x))}$.
- $\frac{d}{dx}a^x = a^x \cdot \ln a$. In the same way, $\frac{d}{dx}e^x = e^x \cdot \ln e = e^x$.
- $\frac{d}{dx}\ln x = \frac{1}{x}$. The chain rule extends this to: $\frac{d}{dx}\ln u = \frac{1}{u} \cdot \frac{du}{dx}$.
- The change-of-base formula states that $\log_a(x) = \frac{\ln x}{\ln a}$. Hence $\frac{d}{dx}log_a(x) = \frac{d}{dx}\frac{\ln x}{\ln a}$. This simplifies to the following rule: $\frac{d}{dx}\log_a(x) = \frac{1}{x \ln a}$.
- The chain rule extends this to $\frac{d}{dx}\log_a u = \frac{1}{u \ln a} \cdot \frac{du}{dx}$.